4. QUATERNION ALGEBRAS

§4.1. Hamilton and His Quaternions

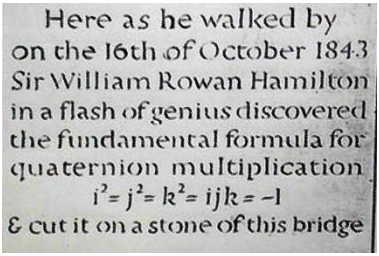
Historically, quaternions were the step between complex numbers and matrices. Hamilton sought in vain to find a 3-dimensional analogue of the way complex numbers represent rotations in 2-dimensional space. His 8 year old son would ask him after breakfast, "Well Papa, can you multiply triplets?" whereupon his father sadly shook his head and said, "no, I can only add and subtract them."

Eventually, in 1843, while walking along beside a canal in Dublin, he realized that he had to consider not triplets but quadruplets, or 'quaternions'. He took out a penknife and carved in Brougham Bridge the key to the problem:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Here i, j, k represent 90° degree rotations about three mutually orthogonal axes.

The other basic relationships:



$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji};$$
$$\mathbf{jk} = \mathbf{i} = -\mathbf{kj};$$
$$\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$$

can be deduced from them, assuming the associative law.

A typical quaternion has the form:

$$x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}.$$

Addition and multiplication are defined in the obvious way, assuming the associative and distributive laws.

Example 1: Writing a typical quaternion as an element $(\lambda, \mathbf{v})$ of $F \times V$, where **i**, **j**, **k** are a basis for V, the operation of multiplication becomes:

$$(\lambda_1, \mathbf{v}_1).(\lambda_2, \mathbf{v}_2) = (\lambda_1\lambda_2 - \mathbf{v}_1.\mathbf{v}_2,\ \lambda_1\mathbf{v}_2 + \lambda_2\mathbf{v}_1 + \mathbf{v}_1\times\mathbf{v}_2).$$

§4.2. Quaternion Algebras

If $a, b \in F^{\#}$ then we define $\mathbf{[}a, b\mathbf{]}_F$ to be a vector space over F of dimension 4 with basis **1**, **i**, **j**, **k** (with F identified with the subspace spanned by **1**) made into an F-algebra by defining multiplication as follows:

	1	**i**	**j**	**k**
1	**1**	**i**	**j**	**k**
i	**i**	**a**	**k**	**–j**
j	**j**	**–k**	**b**	**i**
k	**k**	**j**	**–i**	**–1**

Example 2:
$[-1, -1]_{\mathbb{R}}$ is Hamilton's quaternion algebra.
$[1, -1]_F \cong M_2(F)$, the algebra of 2×2 matrices over F, for any field F.

Here $\mathbf{1} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{i} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{j} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{k} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

§4.3. Quaternion Algebras and Quadratic Forms

If $\mathbf{x} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is an element of the quaternion algebra, A, then the **conjugate** of **x** is defined by: $\overline{\mathbf{x}} = x_0 - x_1\mathrm{i} - x_2\mathrm{j} - x_3\mathrm{k}$.

We define **x** to be a **pure quaternion** if $x_0 = 0$, that is, if:

$$\overline{\mathbf{x}} = -\mathbf{x}.$$

Notation: $\mathbf{A_0}$ denotes the set of pure quaternions in A.

We make A into a quadratic space by defining:

$$\langle \mathbf{x} \mid \mathbf{y} \rangle = \tfrac{1}{2}\,(\mathbf{x}\overline{\mathbf{y}} + \mathbf{y}\overline{\mathbf{x}}).$$

Note that F and A_0 are orthogonal complements of one another and so $A = F \oplus A_0$ as quadratic spaces.

Theorem 1: If $A = [a, b]_F$ then $A \cong \langle 1, -a, -b, ab \rangle$, $F \cong \langle 1 \rangle$ and $A_0 \cong \langle -a, -b, ab \rangle$.
Proof: Take the basis **1**, **i**, **j**, **k**. ✋☺
Corollary: $\det A \cong 1$.

Theorem 2: $[a_1, a_2]_F \cong [b_1, b_2]_F$ as F-algebras if and only if $\langle -a_1, -a_2, a_1a_2\rangle \cong \langle -b_1, -b_2, b_1b_2\rangle$.
Proof: Let $A = [a_1, a_2]_F$ and $B = [b_1, b_2]_F$.
Let $\varphi: A \to B$ be an F-isomorphism.

(1) $\varphi(A_0) = B_0$:
It is sufficient to show that $\varphi(\mathbf{i}), \varphi(\mathbf{j}), \varphi(\mathbf{k}) \in B_0$.
Suppose $\varphi(\mathbf{i}) = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$.
Then $a_1 = a_1\varphi(1) = \varphi(a_1\mathbf{1}) = \varphi(\mathbf{i}^2) = \varphi(\mathbf{i})^2$
$= (x_0^2 + b_1x_1^2 + b_2x_2^2 - b_1b_2x_3^2) + 2x_0(x_1i + x_2j + x_3k)$.
Equating pure parts, $x_0(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}) = 0$.
If $x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} = 0$ then $\varphi(\mathbf{i}) = x_0\mathbf{1} = \varphi(x_0\mathbf{1})$, a contradiction since φ is 1-1.
Hence $x_0 = 0$ and so $\varphi(\mathbf{i}) \in B_0$.
Similarly for $\varphi(\mathbf{j})$ and $\varphi(\mathbf{k})$.

(2) $\overline{\varphi(x)} = \varphi(\overline{x})$: Let $x = y + z$ where $y \in F$ and $z \in A_0$.
Then $\overline{\varphi(x)} = \overline{\varphi(y) + \varphi(z)}$
$= \varphi(y) - \varphi(z) = \varphi(y - z) = \varphi(\overline{x})$.

(3) φ is an isometry:
$\langle\varphi(x) \mid \varphi(x)\rangle = \varphi(x)\,\overline{\varphi(x)} = \varphi(x)\,\varphi(\overline{x}) = \varphi(x\overline{x}) = \langle x \mid x\rangle$,
Since $x\overline{x} \in F$.

Hence A_0, B_0 are isomorphic as quadratic spaces.

Now suppose that $A_0 \cong B_0$.

Then $\langle -a_1, -a_2, a_1a_2\rangle \cong \langle -b_1, -b_2, b_1b_2\rangle$.
Let $\varphi: A_0 \to B_0$ be an isometry.
Then $-\varphi(\mathbf{i})^2 = \varphi(\mathbf{i})\,\varphi(\mathbf{i})$
$= \langle \varphi(\mathbf{i}) \mid \varphi(\mathbf{i})\rangle$
$= \langle \mathbf{i} \mid \mathbf{i}\rangle = -\mathbf{i}^2 = -a_1$.
Hence $\varphi(\mathbf{i})^2 = a_1\mathbf{1}$.
Similarly $\varphi(\mathbf{j})^2 = a_2\mathbf{1}$ and $\varphi(\mathbf{i})\varphi(\mathbf{j}) = -\varphi(\mathbf{j})\varphi(\mathbf{i})$.
Since $\mathbf{1}$, $\varphi(\mathbf{i})$, $\varphi(\mathbf{j})$, $\varphi(\mathbf{k})$ is a basis for B then $B \cong [a_1, a_2]_F$ as F-algebras. ✋☺
Corollary: Quaternion algebras are isomorphic if and only if they are isometric as quadratic spaces.
Proof: This follows from the fact that $A \cong B$ if and only if $A_0 \cong B_0$ (using the Witt Uniqueness Theorem).

Theorem 3: Either $[a, b]_F$ is a division ring or it is isomorphic to $M_2(F)$.
Proof: Suppose $A = [a, b]_F$ is not a division ring.

(1) A is isotropic as a quadratic space:
There exists $0 \neq x \in A$ with no multiplicative inverse.
Now if $\mathbf{x}\bar{\mathbf{x}} \neq 0$ then $\mathbf{x}\left(\frac{\bar{x}}{x\bar{x}}\right) = 1$, a contradiction.
Hence $\langle \mathbf{x} \mid \mathbf{x}\rangle = \mathbf{x}\bar{\mathbf{x}} = 0$.

(2) A is hyperbolic as a quadratic space:
By Theorem 8 of Chapter 2,
$A \cong \langle 1, -1\rangle \oplus \langle c, d\rangle$ for some c, d
$\cong \langle 1, -1\rangle \oplus \langle 1, -1\rangle$ by Theorem 5 of Chapter 2.

(3) A_0 is isotropic as a quadratic space:
A contains two linearly independent elements $x + \mathbf{x}_0$ and $y + \mathbf{y}_0$, with $x, y \in F$ and
$\mathbf{x}_0, \mathbf{y}_0 \in A_0$ which are orthogonal and have zero length.
We may assume without loss of generality that $x = y = 1$.
(If x or $y = 0$ we are done, otherwise we may divide.)
Clearly $\mathbf{x}_0 \neq \mathbf{y}_0$.
From $\langle \mathbf{1} + \mathbf{x}_0 \mid \mathbf{1} + \mathbf{x}_0 \rangle = \langle \mathbf{1} + \mathbf{x}_0 \mid \mathbf{1} + \mathbf{y}_0 \rangle$
$= \langle \mathbf{1} + \mathbf{y}_0 \mid \mathbf{1} + \mathbf{y}_0 \rangle = 0$
we conclude that $\langle \mathbf{x}_0 \mid \mathbf{x}_0 \rangle = \langle \mathbf{x}_0 \mid \mathbf{y}_0 \rangle = \langle \mathbf{y}_0 \mid \mathbf{y}_0 \rangle = -1$
and hence $\langle \mathbf{x}_0 - \mathbf{y}_0 \mid \mathbf{x}_0 - \mathbf{y}_0 \rangle = 0$.

(4) $A \cong M_2(F)$ as F-algebras:
By Theorem 8 of Chapter 2,
$$A_0 \cong \langle -a, -b, ab \rangle \cong \langle 1, -1 \rangle \oplus \langle -1 \rangle.$$
Hence by Theorem 2 above, $A_0 \cong [1, -1]_F \cong M_2(F)$. ✋☺

Example 3: Over $\mathbb{C}$ the only possible quaternion algebra is $M_2(\mathbb{C})$.

Example 4: Over $\mathbb{R}$ the possible quaternion algebras are:

Quaternion algebra	**As a QS**	**Isomorphic to**
$[1, 1]_{\mathbb{R}}$	$\langle 1, -1, -1, 1 \rangle$	$M_2(\mathbf{R})$
$[1, -1]_{\mathbb{R}}$	$\langle 1, -1, 1, -1 \rangle$	$M_2(\mathbf{R})$
$[-1, -1]_{\mathbb{R}}$	$\langle 1, 1, 1, 1 \rangle$	Hamilton's quaternion algebra

Example 5: There are infinitely many Quaternion algebras over $\mathbb{Q}$. In fact, if p, q are distinct primes of the form $4n + 3$ then $[-1, p]_{\mathbb{Q}}$ is not isomorphic to $[-1, q]_{\mathbb{Q}}$. Dirichlet's Theorem from Number Theory ensures that there are infinitely many such primes.

§4.4. The Witt Ring of a Finite Field

Theorem 4: There is only one quaternion algebra over a finite field F, namely $M_2(F)$.

Proof: If F is a finite field and Q is a quaternion algebra over F then $|Q| = |F|^4 < \infty$.

By a theorem of Wedderburn every finite division ring is a field. Since Q is non-commutative it must be isomorphic to $M_2(F)$. ✋☺

Theorem 5: If there is only one quaternion algebra over the field F then

$$W(F) = \{\langle\ \rangle\} + \{\langle x\rangle \mid x \in F^{\#}/F^{\#2}\} + \{\langle 1, x\rangle \mid x \in F^{\#}/F^{\#2}, x \neq -F^{\#2}\}.$$

Addition and multiplication is defined by:

+	**0**	$\langle x\rangle$	$\langle 1, x\rangle$
0	0	$\langle x\rangle$	$\langle 1, x\rangle$
$\langle y\rangle$	$\langle y\rangle$	$\langle 1, xy\rangle$ if $x \neq -y$ 0 if $x = -y$	$\langle -xy\rangle$
$\langle 1, y\rangle$	$\langle 1, y\rangle$	$\langle -xy\rangle$	$\langle 1, -xy\rangle$ if $x \neq y$ 0 if $x = y$

×	**0**	$\langle x \rangle$	$\langle 1, x \rangle$
0	0	0	0
$\langle y \rangle$	0	$\langle xy \rangle$	$\langle 1, x \rangle$
$\langle 1, y \rangle$	0	$\langle 1, y \rangle$	0

Proof: Let $x, y, z \in F^{\#}$.

Putting $a_1 = -\frac{1}{yz}$, $a_2 = -\frac{1}{xz}$, $b_1 = b_2 = 1$ in Theorem 2 we conclude that $\langle 1/yz, 1/xz, 1/xy \rangle \cong \langle -1, -1, 1 \rangle$

$$\cong \langle -1 \rangle \oplus \mathcal{H}.$$

Multiplying by xyz, $\langle x, y, z \rangle \cong \langle -xyz \rangle \oplus \mathcal{H}$.

Hence every non-isotropic quadratic form has degree ≤ 2.

Now, putting $z = -1$ we conclude that:

$$\langle x, y, -1 \rangle \cong \langle xy, 1, -1 \rangle$$

whence, by Witt's Cancellation Theorem, $\langle x, y \rangle \cong \langle 1, xy \rangle$.

Hence every element of $W(F)$ can be written in the form stated.

The addition and multiplication tables can be easily checked. ✋☺

Corollary: Suppose there is only one quaternion algebra over F.

If $-1 \notin F^{\#2}$ then W(F) has exponent 4.

If $-1 \in F^{\#2}$ then W(F) has exponent 2.

Proof: Every element of the form $\langle 1, x \rangle$ has order 2.

So $\langle x \rangle \oplus \langle x \rangle \cong \langle 1, 1 \rangle$.

Hence $\langle x \rangle$ has order $\begin{cases} 4 \text{ if } -1 \notin F^{\#2} \\ 2 \text{ if } -1 \in F^{\#2} \end{cases}$.

Theorem 5: If F is a finite field of odd characteristic,

$$|F^{\#}/F^{\#2}| = 2.$$

Proof: $\{\pm x\} \leftrightarrow x^2$ is a 1-1 correspondence. ✋☺

Theorem 6: If F is a finite field, $|W(F)| = 4$ and

$$W(F) \cong \begin{cases} \mathbb{Z}_4 \text{ if } -1 \notin F^{\#2} \\ \mathbb{Z}_2(C_2) \text{ if } -1 \in F^{\#2} \end{cases}.$$

Proof: If $-1 \notin F^{\#2}$, $W(F) = \{\langle\ \rangle,\ \langle 1\rangle,\ \langle -1\rangle,\ \langle 1, 1\rangle\} \cong \mathbb{Z}_4$.
If $-1 \in F^{\#2}$ and $s \notin F^{\#2}$, $W(F) = \{\langle\ \rangle,\ \langle 1\rangle,\ \langle s\rangle,\ \langle 1, s\rangle\}$
$\cong \mathbb{Z}_2(C_2)$. ✋☺